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A Priori Error Analysis of an Euler Implicit, Finite Element Approximation of the Unsteady Darcy Problem in an Axisymmetric Domain

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Abstract. We consider the time dependent Darcy problem in a three-dimensional axisymmetric domain and, by writing the Fourier expansion of its solution with respect to the angular variable, we observe that each Fourier coefficient satisfies a system of equations on the meridian domain. We propose a discretization of these equations in the case of general solution. This discretization relies on a backward Euler's scheme for the time variable and finite elements for the space variables. We prove a priori error estimates both for the time steps and the meshes.

AMS subject classifications: 65M60, 65M60, 65M15, 76D07, 76M10, 76S05

Key words: Darcy's equations, axisymmetric domain, Fourier truncation, finite element discretization.

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1 Introduction

Let $\breve{\Omega}$ be a bounded three-dimensional domain which is invariant by rotation around an axis. The boundary $\breve{\Gamma}$ of this domain is divided into two parts $\breve{\Gamma}_p$ and $\breve{\Gamma}_u$. We are interested in the following model, suggested by Rajagobal [10],

$$\begin{cases} \partial_t \breve{\boldsymbol{u}} + \alpha \breve{\boldsymbol{u}} + \mathbf{grad} \breve{p} = \breve{\boldsymbol{f}} & \text{in } \breve{\Omega} \times [0,T], \\ \mathrm{div} \breve{\boldsymbol{u}} = 0 & \text{in } \breve{\Omega} \times [0,T], \\ \breve{p} = \breve{p}_b & \text{on } \breve{\Gamma}_p \times [0,T], \\ \breve{\boldsymbol{u}} \cdot \breve{\boldsymbol{n}} = \breve{g} & \text{on } \breve{\Gamma}_u \times [0,T], \\ \breve{\boldsymbol{u}} = \breve{\boldsymbol{u}}_0 & \text{in } \breve{\Omega} \text{ at time } t = 0, \end{cases} \tag{1.1}$$

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where the unknowns are the velocity $\breve{\boldsymbol{u}}$ and the pressure $\breve{p}$ of the fluid. The data are the quantities $\breve{\boldsymbol{f}}$, $\breve{g}$, the pressure on the boundary $\breve{p}_b$ and the initial value of the velocity $\breve{\boldsymbol{u}}_0$. The parameter α is a positive constant representing the drag coefficient. If the problem is set in a domain which is symmetric by rotation around an axis, it is proved in [3] that, when using the Fourier expansion with respect to the angular variable, a three-dimensional problem is equivalent to a system of two-dimensional problems on the meridian domain, each problem being satisfied by a Fourier coefficient of the solution. Here we are going to present the unsteady Darcy equations in three-dimensional axisymmetric geometries, and we propose a discretization of this problem in the case of a general solution, i.e., for the Fourier coefficient of order k, $k\in\mathbb{Z}$. We recall that the same problem are considered in [6] but in the case of an axisymmetric solution, i.e., only for the Fourier coefficient of order $k=0$.

In this work, we assume that the boundary conditions and the external forces are in general case. Axisymmetric problems without any assumption on the data can be transformed into problems which are invariant by rotation see [2, Chap I, prop 1.2.8]. A natural way for reducing axisymmetric problems on $\breve{\Omega}$ to a family of problems on the meridian domain Ω, which we will make precise later, relies on the use of Fourier expansions with respect to the angular variable θ. Then, by using cylindrical coordinates, we can write a variational formulation of this problem in Ω. We prove the well-posedness and some regularity properties of the solution for such a system in the appropriate weighted Sobolev spaces. Next, we propose a time semi-discrete problem that relies on the backward Euler's scheme. We prove that this problem has a unique solution and derive error estimates. Concerning the space discretization, we consider a conforming finite element method which leads to a well-posed discrete problem for which we prove a priori error estimates.

An outline of the paper is as follows:

- In Section 2, we write a variational formulation of problem (1.1) in the case of an axisymmetric domain, we prove its well-posedness and the error issued from Fourier truncation.
- Section 3 is devoted to the description and a priori analysis of the discrete problem in the meridian domain Ω.
- In Section 4, we present some numerical experiments.

2 The two-dimensional problems

Let (x,y,z) denotes a set of Cartesian coordinates in $\mathbb{R}^3$ such that $\breve{\Omega}$ is invariant by rotation around the axis $x=y=0$. We introduce the system of cylindrical coordinates (r,θ,z), with $r\geq 0$ and $-\pi\leq\theta<\pi$, defined by $x=r\cos\theta$ and $y=r\sin\theta$. If Γ_0 denotes the intersection between $\breve{\Omega}$ and axis $r=0$, then there exists a meridian domain Ω in $\mathbb{R}_+\times\mathbb{R}$ such that

$$\breve{\Omega}=\{(r,\theta,z);\ (r,z)\in\Omega\cup\Gamma_0\ \text{ and }\ -\pi<\theta\leq\pi\}.$$

For simplicity, we assume that Γ_0 is the union of a finite number of segments with positive measure. The two-dimensional axisymmetric boundary $\breve{\Gamma}$ of the physical domain $\breve{\Omega}$ is a Lipschitz-continuous boundary and is divided into two parts $\breve{\Gamma}_p$ and $\breve{\Gamma}_u$, also with Lipschitz continous boundaries. The part of the boundary $\breve{\Gamma}_p$ has a positive surface measure. $\breve{\Gamma}_u = \breve{\Gamma}\backslash\overline{\breve{\Gamma}}_p$ is the union of a finite number of surface elements. Setting $\Gamma = \partial\Omega\backslash\Gamma_0$ and rotating Γ around the axis $r=0$ gives back $\breve{\Gamma}$, and Γ_0 is a kind of artificial boundary. We also introduce the two parts Γ_p and $\Gamma_u = \Gamma\backslash\overline{\Gamma}_p$ of the boundary Γ. The unit outward normal vector $\breve{\boldsymbol{n}}$ on $\breve{\Gamma}$ is obtained by rotating the unit outward vector $\boldsymbol{n}$ on Γ.

Each solution of the Darcy equations admits a Fourier expansion with respect to the angular variable θ.

2.1 Fourier expansion

For any function $\breve{v}$ defined on $\breve{\Omega}$, we associate the Fourier coefficients of the corresponding function v on Ω, defined for any k in $\mathbb{Z}$ by

$$v^k(r,z) = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} v(r,\theta,z)e^{-ik\theta}d\theta, \quad v(r,\theta,z) = \frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}} v^k(r,z)e^{ik\theta}.$$

We also introduce the k-dependent operators $\mathbf{grad}_k p$ and $\mathrm{div_k}\boldsymbol{v}$ defined respectively on scalar functions p and on vector fields $\boldsymbol{v}$ by

$$2\mathbf{grad}_k p = \Big(\partial_r p, \frac{ik}{r}p, \partial_z p\Big) \quad \text{and} \quad \mathrm{div}_k \boldsymbol{v} = \partial_r v_r + \frac{1}{r}v_r + \frac{ik}{r}v_\theta + \partial_z v_z.$$

It is checked in [2, IX.1] that $(\breve{\boldsymbol{u}},\breve{p})$ is a solution of problem (1.1) if and only if the pairs $(\boldsymbol{u}_k,p_k)$, $k\in\mathbb{Z}$, are a solution of the system of two-dimensional problems

$$\begin{cases} \partial_t \boldsymbol{u}^k + \alpha\boldsymbol{u}^k + \mathbf{grad}_k p^k = \boldsymbol{f}^k & \text{in } \Omega\times[0,T], \\ \mathrm{div}_k \boldsymbol{u}^k = 0 & \text{in } \Omega\times[0,T], \\ p^k = p_b^k & \text{on } \Gamma_p\times[0,T], \\ \boldsymbol{u}^k\cdot\boldsymbol{n} = g^k & \text{on } \Gamma_u\times[0,T], \\ \boldsymbol{u}^k = \boldsymbol{u}_0^k & \text{in } \Omega \text{ at } t=0. \end{cases} \tag{2.1}$$

We now describe the weighted Sobolev spaces which are needed for the variational formulations of these problems, next we write these formulations and we proove their well-posedness.

2.2 The weighted Sobolev spaces

According to [3, Section II.2], we introduce the spaces (note that the use of Fourier expansions leads to complex-valued functions)

$$L^2_{\pm 1}(\Omega) = \left\{ v:\Omega\to\mathbb{C} \text{ measurable; } \int_\Omega |v(r,z)|^2 r^{\pm 1} drdz < +\infty \right\}.$$

Next, we consider the following spaces

$$H_1^1(\Omega)=\{v\in L_1^2(\Omega);\ \partial_r v\in L_1^2(\Omega)\ \text{ and }\ \partial_z v\in L_1^2(\Omega)\},$$
$$H_{1\diamond}^1(\Omega)=\Big\{q\in H_1^1(\Omega);\ q=0\ \text{ on }\ \Gamma_p\Big\}.$$

We also need the spaces $V_1^1(\Omega)$ and $V_{1\diamond}^1(\Omega)$ defined by

$$V_1^1(\Omega)=H_1^1(\Omega)\cap L_{-1}^2(\Omega),\ \ V_{1\diamond}^1(\Omega)=V_1^1(\Omega)\cap H_{1\diamond}^1(\Omega).$$

All these spaces are provided with the norms which result from their definitions.

For any $k\in\mathbb{Z}$, we denote by $H_{(k)}^1(\Omega)$ the following spaces

$$H_{(k)}^1(\Omega)=\begin{cases} H_1^1(\Omega) & \text{if } k=0,\\ V_1^1(\Omega) & \text{if } |k|\geq 1,\end{cases}$$

provided with its norms and seminorms

$$\|v\|_{H_{(k)}^1(\Omega)}=\Big(\|v\|^2_{H_1^1(\Omega)}+|k|^2\|v\|^2_{L^2_{-1}(\Omega)}\Big)^{\frac12}\quad\text{and}\quad |v|_{H_{(k)}^1(\Omega)}=\Big(|v|^2_{H_1^1(\Omega)}+|k|^2\|v\|^2_{L^2_{-1}(\Omega)}\Big)^{\frac12}.$$

We also define their subspaces $H_{(k)\diamond}^1(\Omega)=H_{(k)}^1(\Omega)\cap H_{1\diamond}^1(\Omega)$. Note that the equivalence of the norm $\|\cdot\|_{H_{(k)}^1(\Omega)}$ and seminorm $|\cdot|_{H_{(k)}^1(\Omega)}$ on $H_{(k)\diamond}^1(\Omega)$, which is obvious for $k\neq 0$, also holds for $k=0$ (see [3, Theorem II.3.1]).

Let us introduce the space $H_{(k)}^{\frac12}(\Gamma_p)$ of traces of functions in $H_{(k)}^1(\Omega)$ on Γ_p. The trace on Γ_u is defined in a nearly standard way see [2, Section 2]. We use the whole scale of Sobolev spaces $H_1^s(\Gamma_u)$, $s\geq 0$, as defined in [3, Chapter II] from

$$L_1^2(\Gamma_u)=\Big\{g:\Gamma_u\to\mathbb{R}\ \text{mesurable};\ \int_{\Gamma_u} g^2(\tau)r(\tau)d\tau<+\infty\Big\},$$

where $r(\tau)$ denotes the distance of the point with tangential coordinate τ to the axis $r=0$. The trace operator: $v\longmapsto v_{|\Gamma_u}$ is continuous and surjective from $H_1^{s+1}(\Omega)$ onto $H_1^{s+\frac12}(\Gamma_u)$, $s\geq 0$, and in particular from $H_1^1(\Omega)$ onto $H_1^{\frac12}(\Gamma_u)$ and also from $V_1^1(\Omega)$ onto the same space $H_1^{\frac12}(\Gamma_u)$, see [3, Chapter II].

2.3 Variational formulation of the two-dimensional problems

We assume that the data $(\boldsymbol{f}^k,p_b^k,g^k)$ belongs to $L^2\big(0,T;L_1^2(\Omega)^3\big)\times L^2\big(0,T;H_{(k)}^{\frac12}(\Gamma_p)\big)\times L^2\big(0,T;L_1^2(\Gamma_u)\big)$ and the datum $\boldsymbol{u}_0^k$ belongs to $L_1^2(\Omega)^3$. Then, the Fourier coefficients $\big((\boldsymbol{u}^k,p^k)\big)_{k\in\mathbb{Z}}$ of a solution of problem (2.1) are a solution of:

Find $(\boldsymbol{u}^k,p^k)$ in $H^1\big(0,T;L^2_1(\Omega)^3\big)\times L^2\big(0,T;H^1_{(k)}(\Omega)\big)$ such that

$$\boldsymbol{u}^k(\cdot,0)=\boldsymbol{u}^k_0 \quad \text{in } \Omega, \tag{2.2a}$$

$$\text{for a.e. } t,\quad 0\le t\le T, \qquad p^k(\cdot,t)=p^k_b \quad \text{on } \Gamma_p, \tag{2.2b}$$

and

$$\forall \boldsymbol{v}\in L^2_1(\Omega)^3, \qquad a_1(\partial_t\boldsymbol{u}^k,\boldsymbol{v})+\alpha a_1(\boldsymbol{u}^k,\boldsymbol{v})+b^k_1(\boldsymbol{v},p^k)=(\boldsymbol{f}^k,\overline{\boldsymbol{v}})_1, \tag{2.3a}$$

$$\forall q\in H^1_{(k)\diamond}(\Omega), \qquad \overline{b^k_1}(\boldsymbol{u}^k,q)=\int_{\Gamma_u} g^k(\tau)q(\tau)r(\tau)d\tau, \tag{2.3b}$$

where the sesquilinear forms $a_1(\cdot,\cdot)$ and $b^k_1(\cdot,\cdot)$ are defined by:

$$a_1(\boldsymbol{u},\boldsymbol{v})=(\boldsymbol{u},\overline{\boldsymbol{v}})_1=\int_\Omega \boldsymbol{u}(r,z)\cdot\overline{\boldsymbol{v}}(r,z)rdrdz,$$

$$b^k_1(\boldsymbol{v},p)=(\overline{\boldsymbol{v}},\mathbf{grad}_k p)_1=\int_\Omega \overline{\boldsymbol{v}}(r,z)\cdot\mathbf{grad}_k p(r,z)rdrdz.$$

Then, the system made by all problems (2.2a)-(2.2b)-(2.3) has a solution $(\boldsymbol{u}^k,p^k)_{k\in\mathbb{Z}}$ in the product space $\Pi_{k\in\mathbb{Z}}H^1\big(0,T;L^2_1(\Omega)^3\big)\times L^2\big(0,T;H^1_{(k)}(\Omega)\big)$.

These forms are obviously continuous on $L^2_1(\Omega)^3\times L^2_1(\Omega)^3$ and $L^2_1(\Omega)^3\times H^1_{(k)}(\Omega)$, respectively. Note that the formula $\overline{b^k_1}(\boldsymbol{v},p)=b^{-k}_1(\overline{\boldsymbol{v}},\overline{p})$ holds and that the kernel

$$\mathbb{V}_k(\Omega)=\Big\{\boldsymbol{v}\in L^2_1(\Omega)^3;\ \forall p\in H^1_{(k)\diamond}(\Omega),\ b^k_1(\boldsymbol{v},p)=0\Big\},$$

is characterized by

$$\mathbb{V}_k(\Omega)=\big\{\boldsymbol{v}\in L^2_1(\Omega)^3;\ \operatorname{div}_k\boldsymbol{v}=0 \ \text{ and } \ \boldsymbol{v}\cdot\boldsymbol{n}=0 \ \text{ on } \Gamma_u\big\}. \tag{2.4}$$

Lemma 2.1. *The pression on $H^1_{(k)}(\Omega)$ vanishes a.e. on the axis Γ_0 in a very weak sense.*

Proof. If $(r_n)_n$ denotes any sequence in $[0,1]$ which tends to 0, then it is a Cauchy sequence and satisfies

$$p^2(r_n,z)-p^2(r_m,z)=2\int_{r_n}^{r_m} p(r,z)(\partial_r p)(r,z)dr.$$

Integrate with respect to z on Γ_0, we obtain

$$\|p(r_n,z)\|^2_{L^2(\Gamma_0)}-\|p(r_m,z)\|^2_{L^2(\Gamma_0)}=2\int_{\Gamma_0}\int_{r_n}^{r_m} p(r,z)(\partial_r p)(r,z)drdz.$$

Using the Cauchy–Schwarz and Young's inequalities, we obtain

$$\begin{aligned}
&\|p(r_n,z)\|^2_{L^2(\Gamma_0)}-\|p(r_m,z)\|^2_{L^2(\Gamma_0)}\\
\le&\Big(\int_{\Gamma_0}\int_{r_n}^{r_m} p^2(r,z)r^{-1}drdz\Big)^{\frac12}\Big(\int_{\Gamma_0}\int_{r_n}^{r_m}(\partial_r p)^2(r,z)rdrdz\Big)^{\frac12}\\
\le&\Big(\int_{\Gamma_0}\int_{r_n}^{r_m} p^2(r,z)r^{-1}drdz+\int_{\Gamma_0}\int_{r_n}^{r_m}(\partial_r p)^2(r,z)rdrdz\Big),
\end{aligned}$$

yields that $\left(\|p(r_n,z)\|^2_{L^2(\Gamma_0)}\right)_n$ is also a Cauchy sequence. Hence, it tends towards a limit. Since p^2 is integrable for the measure $r^{-1}drdz$, this limit is equal to 0, so that the function p vanishes in $r=0$. However the trace is defined in a very weak sense and should be cautiously used. $\square$

As standard for saddle-point problems (see [7, Chapter I, Theorem 4.1]), the well-posedness of problem (2.2a)-(2.2b)-(2.3) relies on the ellipticity of $a_1(\cdot,\cdot)$ and on an inf-sup condition of Babuška and Brezzi type on the form $b_1^k(\cdot,\cdot)$.

Lemma 2.2. *There exists a constant $\beta>0$ independant of k such that the following inf-sup condition holds*

$$\forall q\in H^1_{(k)\diamond}(\Omega),\qquad \sup_{v\in L^2_1(\Omega)^3}\frac{b_1^k(\boldsymbol{v},q)}{\|\boldsymbol{v}\|_{L^2_1(\Omega)^3}}\geq\beta|q|_{H^1_{(k)}(\Omega)}. \tag{2.5}$$

Lemma 2.3. *There exists a constant α^1 independent of k such that, the following ellipticity property holds*

$$\forall\boldsymbol{u}\in L^2_1(\Omega)^3,\quad a_1(\boldsymbol{u},\boldsymbol{u})\geq\alpha^1\|\boldsymbol{u}\|^2_{L^2_1(\Omega)^3}.$$

We refer to [4, Theorem 2.4], for the detailed proof of the next Theorem.

Theorem 2.1. *For any data $\boldsymbol{u}_0^k\in L^2_1(\Omega)^3$, $\boldsymbol{f}^k\in L^2(0,T;L^2_1(\Omega)^3)$, $p_b^k\in L^2(0,T;H^{\frac{1}{2}}_{(k)}(\Gamma_p))$ and $g^k\in L^2(0,T;L^2_1(\Gamma_u))$, the unique solution $\left(\boldsymbol{u}^k,p^k\right)$ of problem* (2.2a)-(2.2b)-(2.3) *belongs to $\in H^1(0,T;L^2_1(\Omega)^3)\times L^2(0,T;H^1_{(k)}(\Omega))$ and satisfies the a priori estimate*

$$\begin{aligned}&\left\|\boldsymbol{u}^k\right\|_{H^1(0,T;L^2_1(\Omega)^3)}+\left\|p^k\right\|_{L^2(0,T;H^1_{(k)}(\Omega))}\\ \leq& c\left(\left\|\boldsymbol{u}_0^k\right\|_{L^2_1(\Omega)^3}+\left\|\boldsymbol{f}^k\right\|_{L^2(0,T;L^2_1(\Omega)^3)}+\left\|p_b^k\right\|_{L^2(0,t;H^{\frac{1}{2}}_{(k)}(\Gamma_p))}+\left\|g^k\right\|_{H^1(0,t;L^2_1(\Gamma_u))}\right).\end{aligned} \tag{2.6}$$

2.4 Fourier truncation

With each $\left(\boldsymbol{f}^k,p_b^k,g^k\right)$ in $L^2(0,T;L^2_1(\Omega)^3)\times L^2(0,T;H^{\frac{1}{2}}_{(k)}(\Gamma_p))\times L^2(0,T;L^2_1(\Gamma_u))$ and $\boldsymbol{u}_0^k\in L^2_1(\Omega)^3$, we associate the unique solution $(\boldsymbol{u}^k,p^k)$ of problem (2.2a)-(2.2b)-(2.3), and we define the three-dimensional functions $\breve{\boldsymbol{u}}$ and $\breve{p}$ by

$$\breve{\boldsymbol{u}}(r,\theta,z)=\frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}}\boldsymbol{u}^k(r,z)e^{ik\theta},\quad \breve{p}(r,\theta,z)=\frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}}p^k(r,z)e^{ik\theta}.$$

It is now readily checked that the corresponding pair $(\breve{\boldsymbol{u}},\breve{p})$ is the only solution of problem (1.1), so that the Darcy problem is fully equivalent to the problem (2.2a)-(2.2b)-(2.3), $k\in\mathbb{Z}$.

Remark 2.1. In the case of axisymmetric data $\breve{\boldsymbol{f}}$, $\breve{p}_b$ and $\breve{g}$, i.e., f_r, f_θ, f_z, p_b and g are independent of θ, all Fourier coefficients of $(\breve{\boldsymbol{u}},\breve{p})$ vanish but those of order zero. We refer to [6] for a slightly different formulation of the problem in this case.

In the case of general data $\breve{\boldsymbol{f}}$, $\breve{p}_b$ and $\breve{g}$, the idea is to solve only a finite number of two-dimensional discrete problems. So, we fix nonnegative integer $\mathcal{K}$, and we introduce the pair $(\breve{\boldsymbol{u}}_\mathcal{K},\breve{p}_\mathcal{K})$ which is obtained from $(\breve{\boldsymbol{u}},\breve{p})$ by Fourier truncation:

$$\breve{\boldsymbol{u}}_\mathcal{K}(r,\theta,z)=\frac{1}{\sqrt{2\pi}}\sum_{|k|\le\mathcal{K}}\boldsymbol{u}^k(r,z)e^{ik\theta},\quad \breve{p}_\mathcal{K}(r,\theta,z)=\frac{1}{\sqrt{2\pi}}\sum_{|k|\le\mathcal{K}}p^k(r,z)e^{ik\theta}. \tag{2.7}$$

We intend to evaluate the distance between $(\breve{\boldsymbol{u}},\breve{p})$ and $(\breve{\boldsymbol{u}}_\mathcal{K},\breve{p}_\mathcal{K})$ in appropriate norms. Estimating this distance relies on some results introduced in [5]. We introduce the spaces $H^{m,s}(\breve{\Omega})$, (see [5, Section II.4.b])

$$H^{m,s}(\breve{\Omega})=\Big\{\breve{v}\in H^m(\breve{\Omega});\ \partial_\theta^\ell\breve{v}\in H^m(\breve{\Omega}),\ 0\le\ell\le s\Big\},$$

and evident extension of spaces defined on each part of $\partial\breve{\Omega}$, where $m\in\mathbb{R}$ and $s\ge 0$. Note that $H^{m,0}(\breve{\Omega})$ coincides with $H^m(\breve{\Omega})$.

From [5, Theorem II.3.1], we obtain the following characterization of $H^{m,s}(\breve{\Omega})$ by Fourier coefficients.

Lemma 2.4. *For any nonnegative real number s and any integer m, the norm*

$$\Big(\sum_{k\in\mathbb{Z}}(1+|k|^2)^s\left\|v^k\right\|^2_{H^m_{(k)}(\Omega)}\Big)^{\frac{1}{2}}$$

is equivalent, on $H^{m,s}(\breve{\Omega})$, to the norm induced by the definition of this space.

Where $H^0_{(k)}(\Omega)=L^2_1(\Omega)$ for $|k|\ge 0$, and $H^1_{(k)}(\Omega)=V^1_1(\Omega)$ for $|k|\ge 1$.

Remark 2.2. When m is a negative integer, we denote by the space $H^m_{(k)\diamond}(\Omega)$ the dual space of $H^{-m}_{(k)}(\Omega)\cap H^{-m}_{1\diamond}(\Omega)$ and provided with the dual norm.

Now, we are in a position to prove the corresponding "anisotropic" regularity result.

Proposition 2.1. For any nonnegative real number s, the mapping which, with data $(\breve{\boldsymbol{f}},\breve{p}_b,\breve{\boldsymbol{u}}_0,\breve{g})$ associates the solution $(\breve{\boldsymbol{u}},\breve{p})$ of problem (1.1) is continuous from :

$$L^2(0,T;H^{0,s}(\breve{\Omega})^3)\times L^2(0,t;H^{\frac{1}{2},s}(\breve{\Gamma}_P))\times H^{0,s}(\breve{\Omega})^3\times H^1(0,t;H^{-1,s}(\breve{\Omega})\cap H^{-1}_{1\diamond}(\breve{\Omega}))$$

into $H^1(0,T;H^{0,s}(\breve{\Omega})^3)\times L^2(0,T;H^{1,s}(\breve{\Omega}))$, i.e.,

$$\begin{aligned}&\|\breve{\boldsymbol{u}}\|_{H^1(0,T;H^{0,s}(\breve{\Omega})^3)}+\|\breve{p}\|_{L^2(0,T;H^{1,s}(\breve{\Omega}))}\\ \le& c\big(\|\breve{\boldsymbol{u}}_0\|_{H^{0,s}(\breve{\Omega})^3}+\left\|\breve{\boldsymbol{f}}\right\|_{L^2(0,T;H^{0,s}(\breve{\Omega})^3)}+\|\breve{p}_b\|_{L^2(0,t;H^{\frac{1}{2},s}(\breve{\Gamma}_P))}+\|\breve{g}\|_{H^1(0,t;H^{-1,s}(\breve{\Omega})\cap H^{-1}_{1\diamond}(\breve{\Omega}))}\big).\end{aligned} \tag{2.8}$$

Proof. From (2.6), we have

$$\left\|\boldsymbol{u}^k\right\|_{H^1(0,T;H^0_{(k)}(\Omega)^3)}+\left\|p^k\right\|_{L^2(0,T;H^1_{(k)}(\Omega))}$$
$$\leq c\left(\left\|\boldsymbol{u}^k_0\right\|_{H^0_{(k)}(\Omega)^3}+\left\|\boldsymbol{f}^k\right\|_{L^2(0,T;H^0_{(k)}(\Omega)^3)}+\left\|p^k_b\right\|_{L^2(0,t;H^{\frac{1}{2}}_{(k)}(\Gamma_p))}+\left\|g^k\right\|_{H^1(0,t;H^{-1}_{(k)\diamond}(\Omega))}\right).$$

Multiplying the square of each term by $(1+|k|^2)^s$ and summing over k with $k\in\mathbb{Z}$, we obtain for any positive integer s

$$\sum_{k\in\mathbb{Z}}(1+|k|^2)^s\left\|\boldsymbol{u}^k\right\|^2_{H^1(0,T;H^0_{(k)}(\Omega)^3)}+\sum_{k\in\mathbb{Z}}(1+|k|^2)^s\left\|p^k\right\|^2_{L^2(0,T;H^1_{(k)}(\Omega))}$$
$$\leq c\Bigg(\sum_{k\in\mathbb{Z}}(1+|k|^2)^s\left\|\boldsymbol{u}^k_0\right\|^2_{H^0_{(k)}(\Omega)^3}+\sum_{k\in\mathbb{Z}}(1+|k|^2)^s\left\|\boldsymbol{f}^k\right\|^2_{L^2(0,T;H^0_{(k)}(\Omega)^3)}$$
$$+\sum_{k\in\mathbb{Z}}(1+|k|^2)^s\left\|p^k_b\right\|^2_{L^2(0,t;H^{\frac{1}{2}}_{(k)}(\Gamma_p))}+\sum_{k\in\mathbb{Z}}(1+|k|^2)^s\left\|g^k\right\|^2_{H^1(0,t;H^{-1}_{(k)\diamond}(\Omega))}\Bigg).$$

Then, Lemma 2.4 yields the desired result. □

Theorem 2.2. *Let s be a nonnegative real number and assume that the data*

$$(\breve{\boldsymbol{f}},\breve{p}_b,\breve{\boldsymbol{u}}_0,\breve{g})\in L^2(0,T;H^{0,s}(\breve{\Omega})^3)\times L^2(0,t;H^{\frac{1}{2},s}(\breve{\Gamma}_P))\times H^{0,s}(\breve{\Omega})^3\times H^1(0,t;H^{-1,s}(\breve{\Omega})\cap H^{-1}_{1\diamond}(\breve{\Omega})).$$

Then, the following bound holds between the solution $(\breve{\boldsymbol{u}},\breve{p})$ of problem (1.1) *and its truncated Fourier series $(\breve{\boldsymbol{u}}_{\mathcal{K}},\breve{p}_{\mathcal{K}})$:*

$$\|\breve{\boldsymbol{u}}-\breve{\boldsymbol{u}}_{\mathcal{K}}\|_{H^1(0,T;L^2(\breve{\Omega})^3)}+\|\breve{p}-\breve{p}_{\mathcal{K}}\|_{L^2\left(0,T;H^1(\breve{\Omega})\right)}$$
$$\leq\mathcal{K}^{-s}\left(\|\breve{\boldsymbol{u}}_0\|_{H^{0,s}(\breve{\Omega})^3}+\left\|\breve{\boldsymbol{f}}\right\|_{L^2(0,T;H^{0,s}(\breve{\Omega})^3)}+\|\breve{p}_b\|_{L^2(0,t;H^{\frac{1}{2},s}(\breve{\Gamma}_p))}+\|\breve{g}\|_{H^1(0,t;H^{-1,s}(\breve{\Omega})\cap H^{-1}_{1\diamond}(\breve{\Omega}))}\right).$$

Proof. From the definition of $\breve{\boldsymbol{u}}_{\mathcal{K}}$ in (2.7), we obtain

$$\|\breve{\boldsymbol{u}}-\breve{\boldsymbol{u}}_{\mathcal{K}}\|^2_{L^2(\breve{\Omega})^3}\leq\sum_{|k|>\mathcal{K}}\left\|\boldsymbol{u}^k\right\|^2_{H^0_{(k)}(\Omega)^3}\leq\sum_{|k|>\mathcal{K}}(1+|k|^2)^s\left\|\boldsymbol{u}^k\right\|^2_{H^0_{(k)}(\Omega)^3}(1+|k|^2)^{-s}.$$

Since $|k|>\mathcal{K}$ and $s\geq 0$, then $(1+|k|^2)^{-s}<\mathcal{K}^{-2s}$ and we have

$$\|\breve{\boldsymbol{u}}-\breve{\boldsymbol{u}}_{\mathcal{K}}\|^2_{L^2(\breve{\Omega})^3}\leq\mathcal{K}^{-2s}\sum_{k\in\mathbb{Z}}(1+|k|^2)^s\left\|\boldsymbol{u}^k\right\|^2_{H^0_{(k)}(\Omega)^3}.$$

When we use Lemma 2.4, we deduce that

$$\|\breve{\boldsymbol{u}}-\breve{\boldsymbol{u}}_{\mathcal{K}}\|^2_{L^2(\breve{\Omega})^3}\leq\mathcal{K}^{-2s}\|\breve{\boldsymbol{u}}\|^2_{H^{0,s}(\Omega)^3}. \tag{2.9}$$

The same arguments lead to

$$\|\partial_t(\breve{\boldsymbol{u}}-\breve{\boldsymbol{u}}_{\mathcal{K}})\|^2_{L^2(\breve{\Omega})^3} \le \mathcal{K}^{-2s}\|\partial_t\breve{\boldsymbol{u}}\|^2_{H^{0,s}(\Omega)^3}, \tag{2.10a}$$

$$\|\breve{p}-\breve{p}_{\mathcal{K}}\|^2_{H^1(\breve{\Omega})} \le \mathcal{K}^{-2s}\|\breve{p}\|^2_{H^{1,s}(\breve{\Omega})}. \tag{2.10b}$$

Integrating (2.9), (2.10a) and (2.10b) between 0 and t, yield

$$\begin{aligned}&\|\breve{\boldsymbol{u}}-\breve{\boldsymbol{u}}_{\mathcal{K}}\|_{H^1(0,T;L^2(\breve{\Omega})^3)}+\|\breve{p}-\breve{p}_{\mathcal{K}}\|_{L^2\left(0,T;H^1(\breve{\Omega})\right)}\\ \le&\mathcal{K}^{-s}\left(\|\breve{\boldsymbol{u}}\|_{H^1(0,T;H^{0,s}(\breve{\Omega})^3)}+\|\breve{p}\|_{L^2\left(0,T;H^{1,s}(\breve{\Omega})\right)}\right).\end{aligned}$$

Finally, from (2.8) we obtain the desired result. □

Remark 2.3. Since the Fourier coefficients of the data $\breve{\boldsymbol{f}}$, $\breve{g}$ and $\breve{p}_b$ cannot be computed explicitly in most practical situations, we introduce the nodes $\theta_m=\frac{2m\pi}{2\mathcal{K}+1}$, $-\mathcal{K}\le m\le\mathcal{K}$, and we define approximate Fourier coefficients by the following formula, for $-\mathcal{K}\le k\le\mathcal{K}$, (same definition of $g^k_*(r,z)$ and $p^k_{b*}(r,z)$)

$$\boldsymbol{f}^k_*(r,z)=\frac{\sqrt{2\pi}}{2\mathcal{K}+1}\sum_{m=-\mathcal{K}}^{\mathcal{K}}\breve{\boldsymbol{f}}(r,\theta_m,z)e^{-ik\theta_m}.$$

3 The discrete problem and its a priori analysis

We split the discretization into two steps: first a semi-discretization in time, and next the full discretization. At each step, we prove a priori error estimates.

3.1 The time semi-discrete problem

We introduce a partition of the interval $[0,T]$ into subintervals $[t_{n-1},t_n]$, $1\le n\le N$, such that $0=t_0<t_1<\cdots<t_N=T$. We denote by τ_n the time step t_n-t_{n-1}, by τ the N- tuple $(\tau_1,\tau_2,\cdots,\tau_N)$ and by $|\tau|$ the maximum of the τ_n, $1\le n\le N$. The time discretization of problem (2.2a)-(2.2b)-(2.3) relies on the use of a backward Euler's scheme. Thus for all $k\in\mathbb{Z}$, for any data $(\boldsymbol{f}^k,p^k_b)\in C^0(0,T;L^2_1(\Omega)^3)\times C^0(0,T;H^{\frac{1}{2}}_{(k)}(\Gamma_p))$, $g^k\in C^0(0,T;L^2_1(\Gamma_u))$ and $\boldsymbol{u}^k_0\in L^2_1(\Omega)^3$, satisfying $\mathrm{div}_k\boldsymbol{u}^k_0=0$ in Ω, we consider the following scheme:

Find $(\boldsymbol{u}^{kn})_{0\le n\le N}\in(L^2_1(\Omega)^3)^{N+1}$ and $(p^{kn})_{1\le n\le N}\in(H^1_{(k)}(\Omega))^N$ such that

$$\boldsymbol{u}^{k0}=\boldsymbol{u}^k_0 \quad\text{in } \Omega, \tag{3.1a}$$

$$\forall n,\ 1\le n\le N,\quad p^{kn}=p^{kn}_0 \quad\text{on } \Gamma_p, \tag{3.1b}$$

$\forall\boldsymbol{v}\in L^2_1(\Omega)^3$ and $\forall q\in H^1_{(k)\diamond}(\Omega)$,

$$(\boldsymbol{u}^{kn},\overline{\boldsymbol{v}})_1+\alpha\tau_n(\boldsymbol{u}^{kn},\overline{\boldsymbol{v}})_1=(\boldsymbol{u}^{k,n-1},\overline{\boldsymbol{v}})_1-\tau_n(\overline{\boldsymbol{v}},\mathbf{grad}_kp^{kn})_1+\tau_n(\boldsymbol{f}^{kn},\overline{\boldsymbol{v}})_1, \tag{3.2a}$$

$$\overline{b^k_1}(\boldsymbol{u}^{kn},q)=\left\langle g^{kn},q\right\rangle_{\Gamma_u}, \tag{3.2b}$$

where $\boldsymbol{f}^{kn}=\boldsymbol{f}^k(\cdot,t_n)$, $g^{kn}=g^k(\cdot,t_n)$ and $p_b^{kn}=p_b^k(\cdot,t_n)$.

We refer to [6, Theorem 3] for the detailed proof of the next Theorem.

Theorem 3.1. *For any data* $(\boldsymbol{f}^k,p_b^k)\in C^0(0,T;L_1^2(\Omega)^3)\times C^0(0,T;H^{\frac12}_{(k)}(\Gamma_p))$, $g^k\in C^0(0,T;L_1^2(\Gamma_u))$ *and* $\boldsymbol{u}_0^k\in L_1^2(\Omega)^3$, *satisfying* $\mathrm{div}_k\boldsymbol{u}_0^k=0$ *in* Ω, *problem* (3.1a)-(3.1b)-(3.2a) *has a unique solution* $(\boldsymbol{u}^{kn},p^{kn})$ *such that* $\forall n$, $0\le n\le N$, $\boldsymbol{u}^{kn}\in L_1^2(\Omega)^3$ *and* $\forall n$, $1\le n\le N$, $p^{kn}\in H^1_{(k)}(\Omega)$.

Moreover the sequences of velocities $(\boldsymbol{u}^{kn})_{0\le n\le N}$ *and pressures* $(p^{kn})_{1\le n\le N}$ *satisfy the following estimates for a constant c independent of n and the time step* τ_n

$$\begin{aligned}\big\|\boldsymbol{u}^{kn}\big\|_{L_1^2(\Omega)^3}\le c\Big(\big\|\boldsymbol{u}_0^k\big\|_{L_1^2(\Omega)^3}+\Big(\sum_{m=1}^n\tau_m\Big(\big\|\boldsymbol{f}^{km}\big\|^2_{L_1^2(\Omega)^3}+\big\|p_b^{km}\big\|^2_{H^{\frac12}_{(k)}(\Gamma_p)}\Big)\Big)^{\frac12}+\big\|g^{kn}\big\|_{L_1^2(\Gamma_u)}\\+\big\|g^k(\cdot,0)\big\|_{L_1^2(\Gamma_u)}\Big),\end{aligned}\tag{3.3a}$$

$$\begin{aligned}\Big(\sum_{m=1}^n\tau_m\Big\|\frac{\boldsymbol{u}^{km}-\boldsymbol{u}^{k,m-1}}{\tau_m}\Big\|^2_{L_1^2(\Omega)^3}\Big)^{\frac12}\le c\Big(\big\|\boldsymbol{u}_0^k\big\|_{L_1^2(\Omega)^3}+\big\|g^k(\cdot,0)\big\|_{L_1^2(\Gamma_u)}\\+\Big(\sum_{m=1}^n\tau_m\Big(\big\|\boldsymbol{f}^{km}\big\|^2_{L_1^2(\Omega)^3}+\big\|p_b^{km}\big\|^2_{H^{\frac12}_{(k)}(\Gamma_p)}\Big)\Big)^{\frac12}\\+\Big(\sum_{m=1}^n\tau_m\Big\|\frac{g^{km}-g^{k,m-1}}{\tau_m}\Big\|^2_{L_1^2(\Gamma_u)}\Big)^{\frac12}\Big),\end{aligned}\tag{3.3b}$$

$$\begin{aligned}\Big(\sum_{m=1}^n\tau_m\big|p^{km}\big|^2_{H^1_{(k)}(\Omega)}\Big)^{\frac12}\le c\Big(\big\|g^k(\cdot,0)\big\|_{L_1^2(\Gamma_u)}+\big\|\boldsymbol{u}_0^k\big\|_{L_1^2(\Omega)^3}\\+\Big(\sum_{m=1}^n\tau_m\Big(\big\|\boldsymbol{f}^{km}\big\|^2_{L_1^2(\Omega)^3}+\big\|p_b^{km}\big\|^2_{H^{\frac12}_{(k)}(\Gamma_p)}+\big\|g^{km}\big\|^2_{L_1^2(\Gamma_u)}\Big)\Big)^{\frac12}\\+\Big(\sum_{m=1}^n\tau_m\Big\|\frac{g^{km}-g^{k,m-1}}{\tau_m}\Big\|^2_{L_1^2(\Gamma_u)}\Big)^{\frac12}\Big).\end{aligned}\tag{3.3c}$$

Remark 3.1. Let Π_τ denote the operator which associates with any continuous function $v\in[0,T]$ the piecewise constant function $\Pi_\tau v$ equal to $v(t_n)$ on each interval $[t_{n-1},t_n]$, $1\le n\le N$. Then, estimate (3.3a) is equivalent to the following

$$\begin{aligned}\sup_{0\le m\le n}\big\|\boldsymbol{u}^{km}\big\|_{L_1^2(\Omega)^3}\le c\Big(\big\|\boldsymbol{u}_0^k\big\|_{L_1^2(\Omega)^3}+\big\|g^k(\cdot,0)\big\|_{L_1^2(\Gamma_u)}+\big\|\Pi_\tau\boldsymbol{f}^k\big\|_{L^2(0,t_n;L_1^2(\Omega)^3)}\\+\big\|\Pi_\tau p_b^k\big\|_{L^2(0,t_n;H^{\frac12}_{(k)}(\Gamma_p))}+\big\|\Pi_\tau g^k\big\|_{L^2(0,t_n;L_1^2(\Gamma_u))}\Big).\end{aligned}$$

In order to state the a priori error estimate, we observe that the family $(\boldsymbol{e}^{kn})_{0\le n\le N}$, with $\boldsymbol{e}^{kn}=\boldsymbol{u}^k(\cdot,t_n)-\boldsymbol{u}^{kn}$ satisfies $\boldsymbol{e}^{k0}=0$ and also, by integrating $\partial_t\boldsymbol{u}^k$ between t_{n-1} and t_n

and subtracting (3.2a) from (2.3) at time t_n,

$$\begin{cases} \forall \boldsymbol{v} \in L_1^2(\Omega)^3, & (\boldsymbol{e}^{kn},\overline{\boldsymbol{v}})_1 + \alpha\tau_n(\boldsymbol{e}^{kn},\overline{\boldsymbol{v}})_1 = (\boldsymbol{e}^{k,n-1},\overline{\boldsymbol{v}})_1 + \tau_n(\epsilon^{kn},\overline{\boldsymbol{v}})_1 \\ & \qquad\qquad -\tau_n b_1^k(\boldsymbol{v},p^k(\cdot,t_n)-p^{kn}), \\ \forall q \in H^1_{(k)\diamond}(\Omega), & \overline{b_1^k}(\boldsymbol{e}^{kn},q)=0, \end{cases} \tag{3.4}$$

where the consistency error ϵ^{kn} is given by

$$\epsilon^{kn} = \frac{1}{\tau_n}\int_{t_{n-1}}^{t_n} (\partial_t \boldsymbol{u}^k)(s)ds - (\partial_t \boldsymbol{u}^k)(t_n).$$

We assume that the velocity $\boldsymbol{u}^k$ of problem (2.2a)-(2.2b)-(2.3) belongs to $H^2\left(0,T;L_1^2(\Omega)^3\right)$, then we can conclude this section, by recalling the main results concerning the a priori estimates, which are proven in [4, Proposition 3.2 and Corollry 3.1], for $1\le n\le N$:

(i) $\left\|\boldsymbol{e}^{kn}\right\|_{L_1^2(\Omega)^3} \le \frac{1}{\sqrt{3}\alpha}|\tau|\left\|\boldsymbol{u}^k\right\|_{H^2(0,t_n;L_1^2(\Omega)^3)}$.

(ii) $\left(\sum_{m=1}^{n}\tau_m\left\|\frac{\boldsymbol{e}^{km}-\boldsymbol{e}^{k,m-1}}{\tau_m}\right\|^2_{L_1^2(\Omega)^3}\right)^{\frac{1}{2}} \le \frac{1}{\sqrt{3}}|\tau|\left\|\boldsymbol{u}^k\right\|_{H^2(0,t_n;L_1^2(\Omega)^3)}$,

$\left(\sum_{m=1}^{n}\tau_m\left|p^k(\cdot,t_m)-p^{km})\right|^2_{H^1_{(k)}(\Omega)}\right)^{\frac{1}{2}} \le \frac{1}{\sqrt{3}}|\tau|\left\|\boldsymbol{u}^k\right\|_{H^2(0,t_n;L_1^2(\Omega)^3)}$.

3.1.1 The three-dimensional error

With each $(\boldsymbol{f}^{kn},p_b^{kn},g^{kn},\boldsymbol{u}_0^k)$, we associate the unique solution $(\boldsymbol{u}^{kn},p^{kn})$ of problem (3.1a)-(3.1b)-(3.2a), and we define the three-dimensional functions $\breve{\boldsymbol{u}}^n$ and $\breve{p}^n$ by

$$\breve{\boldsymbol{u}}^n(r,\theta,z) = \frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}}\boldsymbol{u}^{kn}(r,z)e^{ik\theta}, \quad \breve{p}^n(r,\theta,z) = \frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}}p^{kn}(r,z)e^{ik\theta}.$$

We fix the nonnegative integer $\mathcal{K}$ and we introduce the pair $(\breve{\boldsymbol{u}}^n_{\mathcal{K}},\breve{p}^n_{\mathcal{K}})$ which is obtained from $(\breve{\boldsymbol{u}}^n,\breve{p}^n)$ by Fourier truncation:

$$\breve{\boldsymbol{u}}^n_{\mathcal{K}}(r,\theta,z) = \frac{1}{\sqrt{2\pi}}\sum_{|k|\le\mathcal{K}}\boldsymbol{u}^{kn}(r,z)e^{ik\theta}, \quad \breve{p}^n_{\mathcal{K}}(r,\theta,z) = \frac{1}{\sqrt{2\pi}}\sum_{|k|\le\mathcal{K}}p^{kn}(r,z)e^{ik\theta}. \tag{3.5}$$

We intend to evaluate the error between $(\breve{\boldsymbol{u}}^n,\breve{p}^n)$ and $(\breve{\boldsymbol{u}}^n_{\mathcal{K}},\breve{p}^n_{\mathcal{K}})$ in appropriate norms.

Proposition 3.1. Let s be a nonnegative real number and assume that the data $(\breve{\boldsymbol{f}},\breve{p}_b,\breve{\boldsymbol{u}}_0,\breve{g})$ belongs to $C^0(0,T;H^{0,s}(\breve{\Omega})^3)\times C^0(0,t;H^{\frac{1}{2},s}(\breve{\Gamma}_P))\times H^{0,s}(\breve{\Omega})^3\times C^0(0,t;H^{-1,s}(\breve{\Omega})\cap H^{-1}_{1\diamond}(\breve{\Omega}))$.

Then the following estimates hold

$$\|\breve{\boldsymbol{u}}^n-\breve{\boldsymbol{u}}^n_{\mathcal{K}}\|_{L^2(\breve{\Omega})^3}\le c\mathcal{K}^{-s}\Big(\|\breve{\boldsymbol{u}}_0\|_{H^{0,s}(\breve{\Omega})^3}+\|\breve{g}(\cdot,0)\|_{H^{-1,s}(\breve{\Omega})\cap H^{-1}_{1\diamond}(\breve{\Omega})} +\|\breve{g}^n\|_{H^{-1,s}(\breve{\Omega})\cap H^{-1}_{1\diamond}(\breve{\Omega})}+\Big(\sum_{m=1}^n\tau_m\big(\Big\|\breve{\boldsymbol{f}}^m\Big\|^2_{H^{0,s}(\breve{\Omega})^3}+\|\breve{p}^m_b\|^2_{H^{\frac{1}{2},s}(\breve{\Gamma}_P)}\big)\Big)^{\frac{1}{2}}\Big), \tag{3.6a}$$

$$\Big(\sum_{m=1}^n\tau_m\|\breve{p}^m-\breve{p}^m_{\mathcal{K}}\|^2_{H^1(\breve{\Omega})}\Big)^{\frac{1}{2}}\le c\mathcal{K}^{-s}\Big(\|\breve{\boldsymbol{u}}_0\|_{H^{0,s}(\breve{\Omega})^3}+\|\breve{g}(\cdot,0)\|_{H^{-1,s}(\breve{\Omega})\cap H^{-1}_{1\diamond}(\breve{\Omega})} +\Big(\sum_{m=1}^n\tau_m\Big(\Big\|\breve{\boldsymbol{f}}^m\Big\|^2_{H^{0,s}(\breve{\Omega})^3}+\|\breve{p}^m_b\|^2_{H^{\frac{1}{2},s}(\breve{\Gamma}_P)}+\|\breve{g}^m\|^2_{H^{-1,s}(\breve{\Omega})\cap H^{-1}_{1\diamond}(\breve{\Omega})}\Big)^{\frac{1}{2}}\Big) +\Big(\sum_{m=1}^n\tau_m\Big\|\frac{\breve{g}^m-\breve{g}^{m-1}}{\tau_m}\Big\|^2_{H^{-1,s}(\breve{\Omega})\cap H^{-1}_{1\diamond}(\breve{\Omega})}\Big)^{\frac{1}{2}}\Big), \tag{3.6b}$$

$$\Big(\sum_{m=1}^n\tau_m\Big\|\frac{(\breve{\boldsymbol{u}}^m-\breve{\boldsymbol{u}}^m_{\mathcal{K}})-(\breve{\boldsymbol{u}}^{m-1}-\breve{\boldsymbol{u}}^{m-1}_{\mathcal{K}})}{\tau_m}\Big\|^2_{L^2(\breve{\Omega})^3}\Big)^{\frac{1}{2}} \le c\mathcal{K}^{-s}\Big(\|\breve{\boldsymbol{u}}_0\|_{H^{0,s}(\breve{\Omega})^3}+\|\breve{g}(\cdot,0)\|_{H^{-1,s}(\breve{\Omega})\cap H^{-1}_{1\diamond}(\breve{\Omega})}+\Big(\sum_{m=1}^n\tau_m\Big(\Big\|\breve{\boldsymbol{f}}^m\Big\|^2_{H^{0,s}(\breve{\Omega})^3}+\|\breve{p}^m_b\|^2_{H^{\frac{1}{2},s}(\breve{\Gamma}_P)}\Big)\Big)^{\frac{1}{2}} +\Big(\sum_{m=1}^n\tau_m\Big\|\frac{\breve{g}^m-\breve{g}^{m-1}}{\tau_m}\Big\|^2_{H^{-1,s}(\breve{\Omega})\cap H^{-1}_{1\diamond}(\breve{\Omega})}\Big)^{\frac{1}{2}}\Big). \tag{3.6c}$$

Proof. From (3.3a), we have

$$\Big\|\boldsymbol{u}^{kn}\Big\|_{H^0_{(k)}(\Omega)^3}\le c\Big(\Big\|\boldsymbol{u}^k_0\Big\|_{H^0_{(k)}(\Omega)^3}+\Big(\sum_{m=1}^n\tau_m\Big(\Big\|\boldsymbol{f}^{km}\Big\|^2_{H^0_{(k)}(\Omega)^3}+\Big\|p^{km}_b\Big\|^2_{H^{\frac{1}{2}}_{(k)}(\Gamma_p)}\Big)\Big)^{\frac{1}{2}} +\Big\|g^{kn}\Big\|_{H^{-1}_{(k)\diamond}(\Omega)}+\Big\|g^k(\cdot,0)\Big\|_{H^{-1}_{(k)\diamond}(\Omega)}\Big).$$

Multiplying the square of each term by $(1+|k|^2)^s$, summing over k with $k\in\mathbb{Z}$, and using Lemma 2.4, we obtain for any positive integer s

$$\|\breve{\boldsymbol{u}}^n\|^2_{H^{0,s}(\breve{\Omega})^3}\le c\Big(\|\breve{\boldsymbol{u}}_0\|^2_{H^{0,s}(\breve{\Omega})^3}+\sum_{m=1}^n\tau_m\Big(\Big\|\breve{\boldsymbol{f}}^m\Big\|^2_{H^{0,s}(\breve{\Omega})^3}+\|\breve{p}^m_b\|^2_{H^{\frac{1}{2},s}(\breve{\Gamma}_P)}\Big) +\|\breve{g}^n\|^2_{H^{-1,s}(\breve{\Omega})\cap H^{-1}_{1\diamond}(\breve{\Omega})}+\|\breve{g}(\cdot,0)\|^2_{H^{-1,s}(\breve{\Omega})\cap H^{-1}_{1\diamond}(\breve{\Omega})}\Big). \tag{3.7}$$

The definition of $\breve{\boldsymbol{u}}^n_{\mathcal{K}}$ in (3.5), yields

$$\|\breve{\boldsymbol{u}}^n-\breve{\boldsymbol{u}}^n_{\mathcal{K}}\|^2_{L^2(\breve{\Omega})^3}\le\sum_{|k|>\mathcal{K}}\Big\|\boldsymbol{u}^{kn}\Big\|^2_{H^0_{(k)}(\Omega)^3}.$$

Since, $(1+|k|^2)^{-s}<\mathcal{K}^{-2s}$ and from Lemma 2.4, we obtain

$$\|\breve{\boldsymbol{u}}^n-\breve{\boldsymbol{u}}^n_{\mathcal{K}}\|_{L^2(\breve{\Omega})^3}\le\mathcal{K}^{-s}\|\breve{\boldsymbol{u}}^n\|_{H^{0,s}(\Omega)^3},$$

then (3.6a) follows from (3.7). To prove estimates (3.6b) and (3.6c), we proceed by the same arguments as previous. □

3.2 The time and space discrete problem

We now describe the space discretization of problem (3.1a)-(3.1b)-(3.2a). For each n, $0\leq n\leq N$, let $(\mathcal{T}_{nh})_h$ be a regular family of triangulations of Ω by closed triangles, in the usual sense that

- for each h, $\overline{\Omega}$ is the union of all elements of $\mathcal{T}_{nh}$,
- both $\overline{\Gamma}_p$ and $\overline{\Gamma}_u$ are the union of whole edges of elements of $\mathcal{T}_{nh}$,
- there exists a constant $\sigma>0$ independent of h, n and T such that, for all T in $\mathcal{T}_{nh}$, $\frac{h_T}{\rho_T}\leq\sigma$, where h_T is the diameter of T, and ρ_T the diameter of its inscribed circle,
- h_n is the maximum of the diameters of the elements of $\mathcal{T}_{nh}$,
- $\mathcal{E}_{nh}$ is the set of all edges e of elements T of $\mathcal{T}_{nh}$,
- $\mathcal{E}^0_{nh}$ is the subset of $\mathcal{E}_{nh}$ which elements are not contained in $\partial\Omega$,
- $\mathcal{V}_{nh}$ is the set of vertices of the elements of $\mathcal{T}_{nh}$,
- $\mathcal{V}^0_{nh}$ is the subset of $\mathcal{V}_{nh}$ which elements are inside Ω,
- $\mathcal{V}^b_{nh}=\mathcal{V}_{nh}\setminus\mathcal{V}^0_{nh}$: is the subset of $\mathcal{V}_{nh}$ made of boundary vertices.

For each triangle T and nonnegative integer ℓ, we denote by $P_\ell(T)$ the space of restrictions to T of polynomials with degree $\leq\ell$. At each time step, the discrete space of velocities is:

$$X_{nh}(\Omega)=\left\{\boldsymbol{v}_h\in L^2_1(\Omega)^3;\ \forall T\in\mathcal{T}_{nh},\ \boldsymbol{v}_h|_T\in P_0(T)^3\right\},$$

its interpolation operator is the orthogonal projection operator Π_{nh} from $L^2_1(\Omega)^3$ onto $X_{nh}(\Omega)$ associated with the scalar product of $L^2_1(\Omega)^3$ and verify, for every $0\leq s\leq 1$

$$\forall\boldsymbol{v}\in H^s_{(k)}(\Omega)^3,\quad \|\boldsymbol{v}-\Pi_{nh}\boldsymbol{v}\|_{L^2_1(\Omega)^3}\leq Ch^s_n\|\boldsymbol{v}\|_{H^s_{(k)}(\Omega)^3}. \tag{3.8}$$

We assume that the pressure is continuous whence the choice of discrete space as proposed in [1]:

$$M_{nh(k)}(\Omega)=\Big\{q_h\in H^1_{(k)}(\Omega);\ \forall T\in\mathcal{T}_{nh},\ q_h|_T\in P_1(T)\Big\},$$

its degrees of freedom are defined at the nodes of $\mathcal{V}_{nh}$ and its interpolation operator i_{nh}: $H^1_{(k)}(\Omega)\to M_{nh(k)}(\Omega)$ is the standard Lagrange interpolation operator at the nodes of $\mathcal{V}_{nh}$ with values in $M_{nh(k)}(\Omega)$ and satisfies, for every $\frac{1}{2}<s\leq 1$

$$\forall q\in H^{s+1}_{(k)}(\Omega),\quad |q-i_{nh}q|_{H^1_{(k)}(\Omega)}\leq Ch^s_n\|q\|_{H^{s+1}_{(k)}(\Omega)}. \tag{3.9}$$

Finally, to approximate functions with zero trace on Γ_p, we set

$$M^0_{nh(k)}(\Omega)=\Big\{q_h\in M_{nh(k)}(\Omega);\ q_h=0\ \text{ on }\ \Gamma_p\Big\}.$$

3.2.1 Variational formulation of the discrete problem

For every data $(\boldsymbol{f}^k,p_b^k)_{k\in\mathbb{Z}}\in C^0(0,T;L^2_1(\Omega)^3)\times C^0(0,T;H^{s+\frac12}_{(k)}(\Gamma_p))$, $s>\frac12$, $g^k\in C^0(0,T;L^2_1(\Gamma_u))$ and $\boldsymbol{u}_0^k\in L^2_1(\Omega)^3$ satisfies $\mathrm{div}_k\boldsymbol{u}_0^k=0$ in Ω, the discrete problem constructed by the Galerkin method from (3.1a)-(3.1b)-(3.2a) reads:

Find $(\boldsymbol{u}_h^{kn})_{0\le n\le N}\in(X_{nh}(\Omega))^{N+1}$ and $(p_h^{kn})_{1\le n\le N}\in(M_{nh(k)}(\Omega))^N$ such that

$$\boldsymbol{u}_h^{k0}=\Pi_{0h}\boldsymbol{u}^{k0}\quad\text{in }\ \Omega, \tag{3.10a}$$

$$\forall n,\quad 1\le n\le N,\quad p_h^{kn}=i_{nh}p_b^{kn}\quad\text{on }\ \Gamma_p, \tag{3.10b}$$

$$\begin{aligned}\forall\boldsymbol{v}_h\in X_{nh}(\Omega),\quad &(\boldsymbol{u}_h^{kn},\overline{\boldsymbol{v}}_h)_1+\alpha\tau_n(\boldsymbol{u}_h^{kn},\overline{\boldsymbol{v}}_h)_1+\tau_n b_1^k(\boldsymbol{v}_h,p_h^{kn})\\ &=(\boldsymbol{u}_h^{k,n-1},\overline{\boldsymbol{v}}_h)_1+\tau_n(\boldsymbol{f}^{kn},\overline{\boldsymbol{v}}_h)_1,\end{aligned} \tag{3.10c}$$

$$\forall q_h\in M^0_{nh(k)}(\Omega),\quad \overline{b_1^k}(\boldsymbol{u}_h^{kn},q_h)=\Big\langle g^{kn},q_h\Big\rangle_{\Gamma_u}. \tag{3.10d}$$

Lemma 3.1. *We have the following inf-sup, for all $k\in\mathbb{Z}$,*

$$\forall q_h\in M^0_{nh(k)}(\Omega),\ \sup_{\boldsymbol{v}_h\in X_{nh}(\Omega)}\frac{b_1^k(\boldsymbol{v}_h,q_h)}{\|\boldsymbol{v}_h\|_{L^2_1(\Omega)^3}}\ge|q_h|_{H^1_{(k)}(\Omega)}. \tag{3.11}$$

Theorem 3.2. *For every data $(\boldsymbol{f}^k,p_b^k)$ in $C^0(0,T;L^2_1(\Omega)^3)\times C^0(0,T;H^{s+\frac12}_{(k)}(\Gamma_p))$, where $s>\frac12$, $g^k\in C^0(0,T;L^2_1(\Gamma_u))$ and $\boldsymbol{u}_0^k\in L^2_1(\Omega)^3$ with $\mathrm{div}_k\boldsymbol{u}_0^k=0$ in Ω, problem* (3.10a)-(3.10b)-(3.10c) *has a unique solution $(\boldsymbol{u}_h^{kn},p_h^{kn})$ such that*

$$\forall n,\quad 0\le n\le N,\quad \boldsymbol{u}_h^{kn}\in X_{nh}(\Omega)\quad\text{and}\quad p_h^{kn}\in M_{nh(k)}(\Omega).$$

Moreover, the sequence $\big((\boldsymbol{u}_h^{kn},p_h^{kn})\big)_{0\le n\le N}$ satisfies the following estimates

$$\begin{aligned}\Big\|\boldsymbol{u}_h^{kn}\Big\|_{L^2_1(\Omega)^3}\le c\Big(&\Big\|\boldsymbol{u}_0^k\Big\|_{L^2_1(\Omega)^3}+\Big(\sum_{m=1}^n\tau_m\Big(\Big\|\boldsymbol{f}^{km}\Big\|^2_{L^2_1(\Omega)^3}+\Big\|p_b^{km}\Big\|^2_{H^{s+\frac12}_{(k)}(\Gamma_p)}\Big)\Big)^{\frac12}\\ &+\Big\|g^k(\cdot,0)\Big\|_{L^2_1(\Gamma_u)}+\Big\|g^{kn}\Big\|_{L^2_1(\Gamma_u)}\Big),\end{aligned} \tag{3.12a}$$

$$\begin{aligned}&\Big(\sum_{m=1}^n\tau_m\Big\|\frac{\boldsymbol{u}_h^{km}-\boldsymbol{u}_h^{k,m-1}}{\tau_m}\Big\|^2_{L^2_1(\Omega)^3}\Big)^{\frac12}\\ &\le c\Big(\Big\|\boldsymbol{u}_0^k\Big\|_{L^2_1(\Omega)^3}+\Big\|g^k(\cdot,0)\Big\|_{L^2_1(\Gamma_u)}+\Big(\sum_{m=1}^n\tau_m\Big(\Big\|\boldsymbol{f}^{km}\Big\|^2_{L^2_1(\Omega)^3}+\Big\|p_b^{km}\Big\|^2_{H^{s+\frac12}_{(k)}(\Gamma_p)}\Big)\Big)^{\frac12}\\ &\quad+\Big(\sum_{m=1}^n\tau_m\Big\|\frac{g^{km}-g^{k,m-1}}{\tau_m}\Big\|^2_{L^2_1(\Gamma_u)}\Big)^{\frac12}\Big),\end{aligned} \tag{3.12b}$$

$$\begin{aligned}
&\Big(\sum_{m=1}^{n}\tau_m\Big|p_h^{km}\Big|^2_{H^1_{(k)}(\Omega)}\Big)^{\frac{1}{2}}\\
\leq& c\Big(\Big\|\boldsymbol{u}_0^k\Big\|_{L^2_1(\Omega)^3}+\Big\|g^k(\cdot,0)\Big\|_{L^2_1(\Gamma_u)}+\Big(\sum_{m=1}^{n}\tau_m\Big(\Big\|\boldsymbol{f}^{km}\Big\|^2_{L^2_1(\Omega)^3}+\Big\|p_b^{km}\Big\|^2_{H^{s+\frac{1}{2}}_{(k)}(\Gamma_p)}+\Big\|g^{km}\Big\|^2_{L^2_1(\Gamma_u)}\Big)\Big)^{\frac{1}{2}}\\
&+\Big(\sum_{m=1}^{n}\tau_m\Big\|\frac{g^{km}-g^{k,m-1}}{\tau_m}\Big\|^2_{L^2_1(\Gamma_u)}\Big)^{\frac{1}{2}}\Big). \qquad (3.12c)
\end{aligned}$$

Proof. Applying respectivelly estimates (3.3a), (3.3b) and (3.3c) to problem (3.10a)-(3.10b)-(3.10c) and using the fact that $\|\Pi_{0h}\boldsymbol{u}_0^k\|_{L^2_1(\Omega)^3}\leq\|\boldsymbol{u}_0^k\|_{L^2_1(\Omega)^3}$, we obtain respectivelly (3.12a), (3.12b) and (3.12c). □

3.2.2 A priori error estimates

To establish error estimates, we insert in the error equation an arbitrary element $q_h^{kn}\in M_{nh(k)}(\Omega)$ and we obtain for $1\leq n\leq N$,

$$\begin{aligned}
\forall \boldsymbol{v}_h\in X_{nh},\qquad &(\Pi_{nh}\boldsymbol{u}^{kn}-\boldsymbol{u}_h^{kn},\overline{\boldsymbol{v}}_h)_1+\alpha\tau_n(\Pi_{nh}\boldsymbol{u}^{kn}-\boldsymbol{u}_h^{kn},\overline{\boldsymbol{v}}_h)_1+\tau_n b_1^k(\boldsymbol{v}_h,q_h^{kn}-p_h^{kn})\\
&=(\boldsymbol{u}^{k,n-1}-\boldsymbol{u}_h^{k,n-1},\overline{\boldsymbol{v}}_h)_1-\tau_n b_1^k(\boldsymbol{v}_h,p^{kn}-q_h^{kn}), \qquad (3.13)
\end{aligned}$$

with the starting value $\Pi_{0h}\boldsymbol{u}_0^k-\boldsymbol{u}_h^{k0}=0$ in Ω. We refer to [6, Propositions 1 and 2], for the detailed proof of the next Propositions.

Proposition 3.2. We suppose that $\boldsymbol{u}_0^k\in H^s_{(k)}(\Omega)^3$ and the solution $(\mathbf{u}^{kn},p^{kn})$ belongs to $H^s_{(k)}(\Omega)^3\times H^{s+1}_{(k)}(\Omega)$, for $\frac{1}{2}<s\leq 1$. Then for all n, $1\leq n\leq N$

$$\begin{aligned}
&\Big\|\mathbf{u}^{kn}-\mathbf{u}_h^{kn}\Big\|_{L^2_1(\Omega)^3}\\
\leq& c\Big(\Big(\sum_{m=1}^{n}\tau_m(h_m)^{2s}\Big\|p^{km}\Big\|^2_{H^{s+1}_{(k)}(\Omega)}\Big)^{\frac{1}{2}}+\sum_{m=0}^{n}(h_m)^s\Big\|\mathbf{u}^{km}\Big\|_{H^s_{(k)}(\Omega)^3}\Big). \qquad (3.14)
\end{aligned}$$

Proposition 3.3. If the assumptions of Proposition 3.2 are satisfied, the following a priori error estimate holds for n, $1\leq n\leq N$,

$$\begin{aligned}
&\Big\|\frac{1}{\tau_n}\Pi_{nh}((\mathbf{u}^{kn}-\mathbf{u}_h^{kn})-(\mathbf{u}^{k,n-1}-\mathbf{u}_h^{k,n-1}))+\mathbf{grad}_k(p^{kn}-p_h^{kn})\Big\|_{L^2_1(\Omega)^3}\\
\leq& c\Big(\Big(\sum_{m=1}^{n}\tau_m(h_m)^{2s}\Big\|p^{km}\Big\|^2_{H^{s+1}_{(k)}(\Omega)}\Big)^{\frac{1}{2}}+\sum_{m=0}^{n}\tau_m(h_m)^s\Big\|\mathbf{u}^{km}\Big\|_{H^s_{(k)}(\Omega)^3}+(h_n)^s\Big\|p^{kn}\Big\|_{H^{s+1}_{(k)}(\Omega)}\Big). \qquad (3.15)
\end{aligned}$$

The next result gives another error estimate where both velocity error and pressure error can be obtained separately. This is very practice in numerical point of view.

Corollary 3.1. *Assume that all elements of $\mathcal{T}_{n-1,h}$ are contained in elements of $\mathcal{T}_{nh}$, $0\le n\le N$. If assumptions of Proposition 3.2 are satisfied, estimate* (3.15) *still holds with its left-hand side replaced by*

$$\left\|\frac{1}{\tau_n}\Pi_{nh}((\boldsymbol{u}^{kn}-\boldsymbol{u}_h^{kn})-(\boldsymbol{u}^{k,n-1}-\boldsymbol{u}_h^{k,n-1}))\right\|_{L^2_1(\Omega)^3}+\left\|\mathbf{grad}_k(p^{kn}-p_h^{kn})\right\|_{L^2_1(\Omega)^3}.$$

3.2.3 The three-dimensional error

Now we will back to the three-dimensional problem. For this, once the discret coefficients $(\boldsymbol{u}_h^{kn},p_h^{kn})$, $|k|\le\mathcal{K}$, are known, the basic idea is to define the three-dimensional discrete solution

$$\breve{\boldsymbol{u}}^n_{\mathcal{K},h}(r,\theta,z)=\frac{1}{\sqrt{2\pi}}\sum_{|k|\le\mathcal{K}}\boldsymbol{u}_h^{kn}(r,z)e^{ik\theta}, \tag{3.16a}$$

$$\breve{p}^n_{\mathcal{K},h}(r,\theta,z)=\frac{1}{\sqrt{2\pi}}\sum_{|k|\le\mathcal{K}}p_h^{kn}(r,z)e^{ik\theta}. \tag{3.16b}$$

Indeed, bounding the error between the solution $(\breve{\boldsymbol{u}}^n,\breve{p}^n)$ and this solution relies on the triangle inequality

$$\left\|\breve{\boldsymbol{u}}^n-\breve{\boldsymbol{u}}^n_{\mathcal{K},h}\right\|_{L^2(\breve{\Omega})^3}\le\|\breve{\boldsymbol{u}}^n-\breve{\boldsymbol{u}}^n_{\mathcal{K}}\|_{L^2(\breve{\Omega})^3}+\left\|\breve{\boldsymbol{u}}^n_{\mathcal{K}}-\breve{\boldsymbol{u}}^n_{\mathcal{K},h}\right\|_{L^2(\breve{\Omega})^3}.$$

The first term in the right-hand side of this inequality is evaluated in Proposition 3.1, while the second one obviously satisfies,

$$\left\|\breve{\boldsymbol{u}}^n_{\mathcal{K}}-\breve{\boldsymbol{u}}^n_{\mathcal{K},h}\right\|^2_{L^2(\breve{\Omega})^3}=\sum_{|k|\le\mathcal{K}}\left\|\boldsymbol{u}^{kn}-\boldsymbol{u}_h^{kn}\right\|^2_{H^0_{(k)}(\Omega)^3}.$$

Since $(1+|k|^2)^s\ge1$, we obtain for any real number s, $\frac{1}{2}<s\le1$

$$\left\|\breve{\boldsymbol{u}}^n_{\mathcal{K}}-\breve{\boldsymbol{u}}^n_{\mathcal{K},h}\right\|^2_{L^2(\breve{\Omega})^3}\le\sum_{k\in\mathbb{Z}}(1+|k|^2)^s\left\|\boldsymbol{u}^{kn}-\boldsymbol{u}_h^{kn}\right\|^2_{H^0_{(k)}(\Omega)^3}$$

and its analogue for $\left\|\breve{p}^n-\breve{p}^n_{\mathcal{K},h}\right\|_{H^1(\breve{\Omega})}$. So the following results are easily derived from Propositions 3.2, 3.3 and Lemma 2.4.

Theorem 3.3. *For any $\frac{1}{2}<s\le1$, assume that the data $(\breve{\boldsymbol{f}},\breve{p}_b,\breve{\boldsymbol{u}}_0,\breve{g})$ belongs to $C^0(0,T;H^{0,s}(\breve{\Omega})^3)\times C^0(0,t;H^{\frac{1}{2},s}(\breve{\Gamma}_p))\times H^{0,s}(\breve{\Omega})^3\times C^0(0,t;H^{-1,s}(\breve{\Omega})\cap H^{-1}_{1\diamond}(\breve{\Omega}))$.*

If assumptions of the Proposition 3.2 are satisfied, the following error estimate holds between *the solution* $(\breve{\boldsymbol{u}}^n,\breve{p}^n)$ *and* $(\breve{\boldsymbol{u}}^n_{\mathcal{K},h},\breve{p}^n_{\mathcal{K},h})$ *defined in* (3.16)*: for* $n, 1\le n\le N$,

$$\begin{aligned}\left\|\breve{\boldsymbol{u}}^n-\breve{\boldsymbol{u}}^n_{\mathcal{K},h}\right\|_{L^2(\breve{\Omega})^3}\le & c\Big(\Big(\sum_{m=1}^n\tau_m(h_m)^{2s}\|\breve{p}^m\|^2_{H^{s+1,s}(\breve{\Omega})}\Big)^{\frac12}+\sum_{m=0}^n(h_m)^s\|\breve{\boldsymbol{u}}^m\|_{H^{s,s}(\breve{\Omega})^3}\\ &+\mathcal{K}^{-s}\Big(\|\breve{\boldsymbol{u}}_0\|_{H^{0,s}(\breve{\Omega})^3}+\|\breve{g}(\cdot,0)\|_{H^{-1,s}(\breve{\Omega})\cap H^{-1}_{1\diamond}(\breve{\Omega})}+\|\breve{g}^n\|_{H^{-1,s}(\breve{\Omega})\cap H^{-1}_{1\diamond}(\breve{\Omega})}\\ &+\Big(\sum_{m=1}^n\tau_m\Big(\Big\|\breve{\boldsymbol{f}}^m\Big\|^2_{H^{0,s}(\breve{\Omega})^3}+\|\breve{p}^m_b\|^2_{H^{\frac12,s}(\breve{\Gamma}_p)}\Big)\Big)^{\frac12}\Big)\Big).\end{aligned}\tag{3.17}$$

Theorem 3.4. *Assume that all elements of* $\mathcal{T}_{n-1,h}$ *are contained in elements of* $\mathcal{T}_{nh}$ *and that for any real number* s, $\frac12<s\le1$, *the data* $(\breve{\boldsymbol{f}},\breve{p}_b,\breve{\boldsymbol{u}}_0,\breve{g})$ *belongs to*

$$C^0(0,T;H^{0,s}(\breve{\Omega})^3)\times C^0(0,t;H^{\frac12,s}(\breve{\Gamma}_P))\times H^{0,s}(\breve{\Omega})^3\times C^0(0,t;H^{-1,s}(\breve{\Omega})\cap H^{-1}_{1\diamond}(\breve{\Omega})).$$

If assumptions of Proposition 3.2 are satisfied, then the following error estimate holds between the solution $(\breve{\boldsymbol{u}}^n,\breve{p}^n)$ *and* $(\breve{\boldsymbol{u}}^n_{\mathcal{K},h},\breve{p}^n_{\mathcal{K},h})$ *defined in* (3.16)*: for* n, $1\le n\le N$,

$$\begin{aligned}&(\tau_n)^{\frac12}\left\|\breve{p}^n-\breve{p}^n_{\mathcal{K},h}\right\|_{H^1(\breve{\Omega})}\\ \le&|\tau|^{\frac12}\Big(\Big(\sum_{m=1}^n\tau_m(h_m)^{2s}\|\breve{p}^m\|^2_{H^{s+1,s}(\breve{\Omega})}\Big)^{\frac12}+h_n\|\breve{p}^n\|_{H^{s+1,s}(\breve{\Omega})}\\ &+\sum_{m=0}^n(h_m)^s\|\breve{\boldsymbol{u}}^m\|_{H^{s,s}(\breve{\Omega})^3}\Big)+\mathcal{K}^{-s}\Big(\|\breve{\boldsymbol{u}}_0\|_{H^{0,s}(\breve{\Omega})^3}+\|\breve{g}(\cdot,0)\|_{H^{-1,s}(\breve{\Omega})\cap H^{-1}_{1\diamond}(\breve{\Omega})}\\ &+\Big(\sum_{m=1}^n\tau_m\Big(\Big\|\breve{\boldsymbol{f}}^m\Big\|^2_{H^{0,s}(\breve{\Omega})^3}+\|\breve{p}^m_b\|^2_{H^{\frac12,s}(\breve{\Gamma}_p)}+\|\breve{g}^n\|_{H^{-1,s}(\breve{\Omega})\cap H^{-1}_{1\diamond}(\breve{\Omega})}\Big)\Big)^{\frac12}\Big)\\ &+\Big(\sum_{m=1}^n\tau_m\Big\|\frac{\breve{g}^m-\breve{g}^{m-1}}{\tau_m}\Big\|^2_{H^{-1,s}(\breve{\Omega})\cap H^{-1}_{1\diamond}(\breve{\Omega})}\Big)\Big)^{\frac12}\Big).\end{aligned}\tag{3.18}$$

4 Numerical experiments

In this section we numerically investigate the approximation of two, cylindrically symmetric Darcy flow problem. These experiments are performed using FreeFEM++-Software [8] on an example with a known solution. Rates of convergence in time and space of the approximation to the known solution are computed and compared with those predicted by theoritical error estimates.

The order of convergence (time or space) is estimated by dividing the errors above, computed for two sets of parameters where the axisymmetric Darcy's equation is computed in the meridian domain $\Omega=(0,\frac12)\times(-\frac12,\frac12)$. For both simulations, $\Gamma_{\boldsymbol{u}}$ coincides with the border $0\times(-\frac12,\frac12)$ and $\Gamma_p=\partial\Omega/\Gamma_{\boldsymbol{u}}$.

4.1 Time accuracy

The first test is used to validate the time accuracy. In order to evaluate the convergence rates, we consider the exact solution given by

$$\begin{aligned} \boldsymbol{u}(r,z) &= \left(rz, \frac{1}{4} - z^2\right)\sin(\pi t), \\ p(r,z) &= rz\cos(\pi t) + (2r+3z)(t^2+t-2) - \frac{2}{3}, \quad 0 \le t \le 1. \end{aligned}$$

Hence, the suitable forcing functions $\boldsymbol{f}$, g and p_b are obtained using this exact solution in Darcy's equation.

Table 1: Time accuracy.

τ	$\|\boldsymbol{u}-\boldsymbol{u}_{app}\|_{L^2_1(\Omega)^3}$	$\mathcal{O}(\boldsymbol{u})$
0.1	0.0081762	—-
0.05	0.00417279	0.97
0.025	0.00210943	0.98
0.0125	0.00106205	0.99
0.00625	0.000535709	0.99

In Table 1, we plotted the L^2_1 error of the velocity, between the numerical solution and the exact solution at final time $T=1$ for different time steps τ. We observe that the order of accuracy in time equal to 1 which is in concordance with a priori error estimate obtained above, when the backward Euler time differentiation is used. A plot of the exact and approximate velocity field $\boldsymbol{u}$, and the pressure p are given in Figs. 1 and 2, respectively.

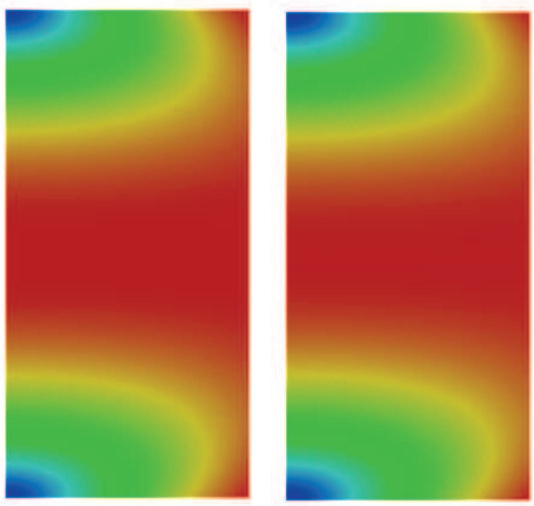

Figure 1: Velocity: Exact in the left and approximate solution in the right.

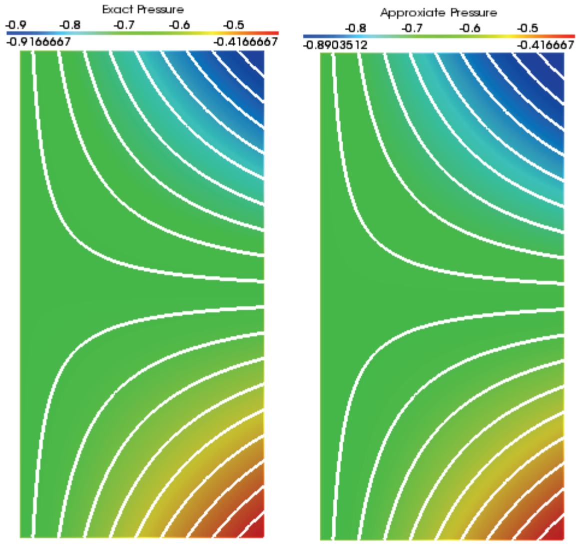

Figure 2: Pressure and isolines: Exact in the left and approximate solution in the right.

4.2 space accuracy

The example considered now is a modification of the Taylor-Green vortex flow problem, a prototypal problem in Navier-Stokes flow approximation. The exact velocity and pressure are given as follow: $(t \in [0,1])$

$$\boldsymbol{u}(r,z) = \Big(t(-r\cos(\pi r)\sin(\pi z)), t\Big(-\frac{2}{\pi}\cos(\pi r)\cos(\pi z) + r\sin(\pi r)\cos(\pi z)\Big)\Big),$$
$$p(r,z) = \sin(\pi z)(-\cos(\pi r) + 2\pi r\sin(\pi r)) + t - 1.$$

To estimate the space convergence rate, we have used several mesh sizes h. We recover a convergence order for velocity-pressure that decreas as h tends to 0, which is confirm theoretical estimate obtained above. Table 2 shwos that the convergnce order is ≈ 1.

Figs. 3, 4 and 5 represent the exact and approximate velocity, velocity field and pressure.

Table 2: Space accuracy.

h	$\|\boldsymbol{u}-\boldsymbol{u}_{app}\|_{L^2_1(\Omega)^3}$	$\|\text{grad}\,(p-p_{app})\|_{L^2_1(\Omega)^3}$	$\mathcal{O}(\boldsymbol{u}) - \mathcal{O}(p)$
0.149071	0.0866377	0.285598	———
0.0786165	0.0331702	0.143125	1.5 – 1.08
0.0408695	0.0152626	0.0710153	1.19 – 1.07
0.0207535	0.0065823	0.0356921	1.24 – 1.05
0.0114654	0.0030579	0.0178266	1.29 – 1.17
0.00557817	0.0015102	0.00891096	0.98 – 0.96

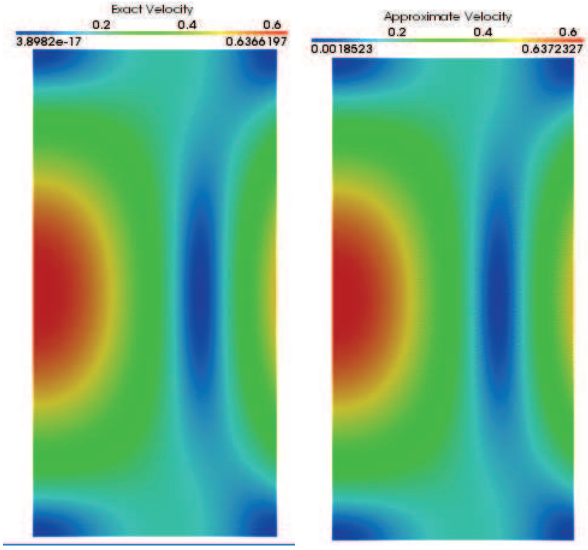

Figure 3: Velocity: Exact in the left and approximate solution in the right.

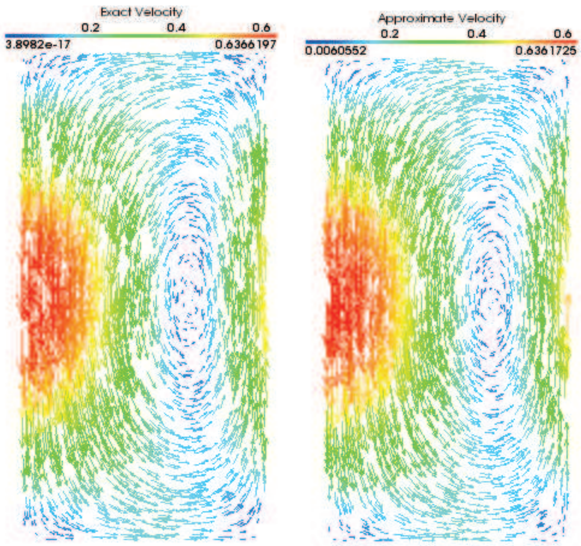

Figure 4: Vector Fields of the Velocity : Exact in the left and approximate solution in the right.

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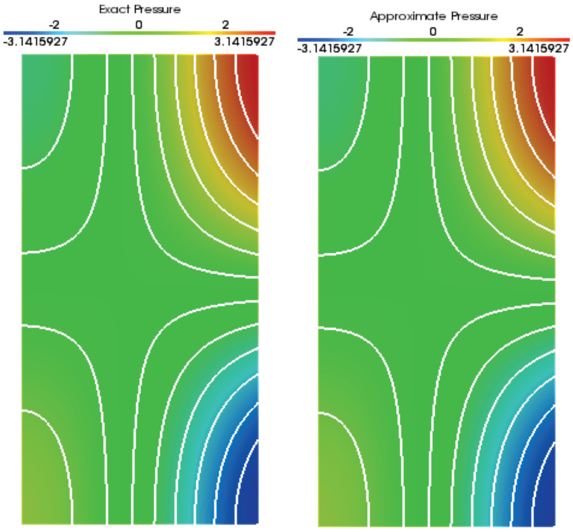

Figure 5: Pressure and isolines: Exact in the left and approximate solution in the right.

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